THE DIRICHLET SERIES FOR THE EXTERIOR SQUARE L-FUNCTION ON $GL(n)$

ALEX V. KONTOROVICH

1. Introduction

Let f be a Hecke-Maass cusp form on $\mathrm{PSL}(n,\mathbb{Z})\backslash \mathrm{PGL}(n,\mathbb{R})/O(n)$, see e.g. [Gol06]. Let $A(m_1, m_2, \ldots, m_{n-1})$ be the Fourier coefficients in its Jacquet-Whittaker expansion. Then the Godement-Jacquet L-function associated to f is given as a Dirichlet series, and in terms of the Langlands-Satake parameters $\alpha_i(p)$, by

$$L(f,s) = \sum_{m\geq 1} \frac{A(m,1,1,\ldots,1)}{m^s} = \prod_p \prod_{i=1}^n \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}.$$

The exterior square L-function is defined via the Euler product

$$L(f,s,\wedge^2) = \prod_p \prod_{1\leq i<j\leq n} \left(1 - \frac{\alpha_i(p)\alpha_j(p)}{p^s}\right)^{-1}. \tag{1}$$

Two distinct representations of the exterior square L-function are known, the first due to Jacquet and Shalika [JS90], and the second discovered by Bump and Friedberg [BF90]. It is our goal in this short note to present an elementary derivation of the Jacquet-Shalika construction, expressing the Euler product in (1) as a classical Dirichlet series in the Fourier coefficients $A(m_1, \ldots, m_{n-1})$.

On GL(2),

$$L(f,s,\wedge^2) = \prod_p \left(1 - \frac{\alpha(p)\bar{\alpha}(p)}{p^s}\right)^{-1} = \zeta(s).$$

On GL(3), it is easy to see that the exterior square L-function

$$L(f,s,\wedge^2) = L(\tilde{f},s) = \sum_m \frac{A(1,m)}{m^s}$$

is just the dual L-function corresponding to the contragredient form $\tilde{f}$.

Date: September 26, 2009.

On GL(4), experts have known for some time that the exterior square L-function can be expressed as a zeta function times the "Middle" L-function:

$$L(f,s,\wedge^2)=\zeta(2s)\sum_m \frac{A(1,m,1)}{m^s}.$$

The general formula on GL(n) is as follows.

Theorem 1. *For odd $n \geq 3$, the Dirichlet series for the exterior square L-function is given by*

$$L(f,s,\wedge^2)=\sum_{m_2,m_4,\ldots,m_{n-1}\geq 1}\frac{A(1,m_2,1,m_4,1,\ldots,1,m_{n-1})}{(m_2\, m_4^2\, m_6^3\cdots m_{n-1}^{(n-1)/2})^s}.$$

For even $n \geq 2$, we have

$$L(f,s,\wedge^2)=\zeta\left(\frac{n}{2}\,s\right)\sum_{m_2,m_4,\ldots,m_{n-2}\geq 1}\frac{A(1,m_2,1,m_4,1,\ldots,1,m_{n-2},1)}{(m_2\, m_4^2\, m_6^3\cdots m_{n-2}^{(n-2)/2})^s}.$$

2. Proof of Theorem 1

As is well-known from the work of Shintani and Casselman-Shalika, the Fourier coefficient $A(p^{k_1},...,p^{k_{n-1}})$ is the Schur function

$$A(p^{k_1},\ldots,p^{k_{n-1}})=S_\lambda(\alpha). \tag{2}$$

Here

$$\alpha=(\alpha_1(p),\ldots,\alpha_n(p))$$

are the Langlands-Satake parameters, and

$$\lambda=(\lambda_1,\ldots,\lambda_n), \qquad \text{where} \qquad \lambda_j=\sum_{i>j}k_i.$$

Recall the following identity [BF90, (3.3)]:

$$\begin{aligned}\sum_{k_1,k_2,\ldots,k_{n-1}\geq 0} & S_\lambda(\alpha)\ X^{k_1+k_3+k_5+\cdots}\ Y^{k_2+k_3+2k_4+2k_5+\cdots}\\ &= L_0\prod_i(1-\alpha_i(p)X)^{-1}\prod_{i<j}(1-\alpha_i(p)\alpha_j(p)Y)^{-1}, \end{aligned}\tag{3}$$

where

$$L_0=\begin{cases}1-Y^{n/2} & \text{if } n \text{ is even;}\\ 1-XY^{(n-1)/2} & \text{if } n \text{ is odd.}\end{cases}$$

Setting $X = p^{-s}$, $Y = p^{-w}$, using (2), and taking the product over all primes p of both sides of (3) gives

$$\mathfrak{Z}(s,w) := \sum_{r_1,r_2,\ldots,r_{n-1}\geq 1} \frac{A(r_1,r_2,\ldots,r_{n-1})}{r_1^s\, r_2^w\, r_3^{s+w}\, r_4^{2w}\, r_5^{s+2w}\cdots} = Z(s,w)L(f,s)L(f,w,\wedge^2),$$

where

$$Z(s,w) = \begin{cases} 1/\zeta(\frac{n}{2}w) & \text{if } n \text{ is even;} \\ 1/\zeta(s+\frac{n-1}{2}w) & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, we have the following Hecke relations [Gol06, Theorem 9.3.11]. For n odd,

$$\begin{aligned} & A(m,1,1,\ldots,1)A(1,m_2,1,m_4,1,\ldots,1,m_{n-1}) \\ = & \sum_{\substack{c_2\,c_4\,c_6\cdots c_{n-1}\,c_n=m \\ c_2|m_2,c_4|m_4,c_6|m_6,\ldots,c_{n-1}|m_{n-1}}} A(c_n, \frac{m_2}{c_2}, c_2, \frac{m_4}{c_4}, c_4, \ldots, \frac{m_{n-1}}{c_{n-1}}). \end{aligned}$$

For n even, we have

$$\begin{aligned} & A(m,1,1,\ldots,1)A(1,m_2,1,m_4,1,\ldots,1,m_{n-2},1) \\ = & \sum_{\substack{c_2\,c_4\,c_6\cdots c_{n-2}\,c_n=m \\ c_2|m_2,c_4|m_4,c_6|m_6,\ldots,c_{n-2}|m_{n-2}}} A(c_n, \frac{m_2}{c_2}, c_2, \frac{m_4}{c_4}, c_4, \ldots, \frac{m_{n-2}}{c_{n-2}}, c_{n-2}). \end{aligned}$$

In either case, dividing both sides by m^s and $(m_2\, m_4^2\, m_6^3\cdots)^w$ and summing gives

$$\begin{aligned} \mathcal{Z}(s,w) &:= \left(\sum_{m\geq 1} \frac{A(m,1,1,\ldots,1)}{m^s}\right)\left(\sum_{m_2,m_4,\cdots\geq 1} \frac{A(1,m_2,1,m_4,1,\ldots)}{(m_2\, m_4^2\, m_6^3\cdots)^w}\right) \\ &= \sum_{m,m_2,m_4,\cdots\geq 1}\ \sum_{\substack{c_2\,c_4\cdots=m \\ c_2|m_2,c_4|m_4,\cdots}} \frac{A(c_n, \frac{m_2}{c_2}, c_2, \frac{m_4}{c_4}, c_4,\ldots)}{m^s(m_2\, m_4^2\, m_6^3\cdots)^w}. \end{aligned}$$

Interchange the orders of summation and write $m_i = m_i' c_i$:

$$\mathcal{Z}(s,w) = \sum_{c_2,c_4,\cdots,c_n\geq 1}\ \sum_{\substack{m_2',m_4',m_6'\cdots\geq 1 \\ m=c_2\,c_4\cdots c_n}} \frac{A(c_n, m_2', c_2, m_4', c_4,\ldots)}{(c_2\,c_4\cdots c_n)^s(m_2'\,c_2\,m_4'^2\,c_4^2\,m_6'^3\,c_6^3\cdots)^w}.$$

Rename $r_1 = c_n$, $r_2 = m_2'$, $r_3 = c_2, \ldots,$ with

$$r_{n-1} = \begin{cases} m_{n-1}' & \text{if } n \text{ is odd;} \\ c_{n-2} & \text{if } n \text{ is even.} \end{cases}$$

If n is even, then

$$\mathcal{Z}(s,w) = \mathfrak{Z}(s,w)$$

and dividing both sides by $L(f,s)$ proves the theorem in this case.

If n is odd, then we have an additional sum over $r_n = c_{n-1}$, which does not appear inside the Fourier coefficients. Thus

$$\mathcal{Z}(s,w) = \zeta(s + \frac{n-1}{2}w)\mathfrak{Z}(s,w) = L(f,s)L(f,w,\wedge^2).$$

Again, dividing both sides by $L(f,s)$ gives the desired result. This completes the proof. $\square$

The referee has kindly pointed out to us that the argument above is equivalent to the fact [Lit40, page 238 (11.9;2)] that

$$\prod_{i<j}(1-\alpha_i\alpha_j)^{-1} = \sum S_\lambda(\alpha),$$

where the summation runs over partitions λ whose conjugate partition is even.

Acknowledgements. The author wishes to express his gratitude to Dorian Goldfeld, Meera Thillainatesan, Sol Friedberg, and the referee for many comments and corrections to an earlier draft.

References

[BF90] Daniel Bump and Solomon Friedberg. The exterior square automorphic L-functions on $\mathrm{GL}(n)$. In *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, volume 3 of *Israel Math. Conf. Proc.*, pages 47–65. Weizmann, Jerusalem, 1990.

[Gol06] Dorian Goldfeld. *Automorphic forms and L-functions for the group* $\mathrm{GL}(n,\mathbf{R})$, volume 99 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. With an appendix by Kevin A. Broughan.

[JS90] Hervé Jacquet and Joseph Shalika. Exterior square L-functions. In *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 143–226. Academic Press, Boston, MA, 1990.

[Lit40] Dudley E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York, 1940.

E-mail address: `alexk@math.brown.edu`

Department of Mathematics, Columbia University, New York, NY, 10027

Current address: Department of Mathematics, Brown University, Providence, RI, 02912